

Fluctuations of the eigenvalue number in the fixed interval for β -models with $\beta = 1, 2, 4$

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This paper is dedicated to Prof. Brunello Tirozzi on the occasion of his 70th birthday.

Abstract

We study the fluctuation of the eigenvalue number of any fixed interval $\Delta = [a, b]$ inside the spectrum for β -ensembles of random matrices in the case $\beta = 1, 2, 4$. We assume that the potential V is polynomial and consider the cases of any multi-cut support of the equilibrium measure. It is shown that fluctuations become gaussian in the limit $n \rightarrow \infty$, if they are normalized by $\pi^{-2} \log n$.

1 Introduction and main results

Consider β -ensemble of random matrices, whose joint eigenvalue distribution is

$$p_{n,\beta}(\lambda_1, \dots, \lambda_n) = Q_{n,\beta}^{-1}[V] \prod_{i=1}^n e^{-n\beta V(\lambda_i)/2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta, \quad (1.1)$$

where $Q_{n,\beta}[V]$ is a normalizing factor

$$Q_{n,\beta}[V] = \int \prod_{i=1}^n e^{-n\beta V(\lambda_i)/2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\bar{\lambda}.$$

The function V (called the potential) is a real valued Hölder function satisfying the condition

$$V(\lambda) \geq 2(1 + \epsilon) \log(1 + |\lambda|). \quad (1.2)$$

Below we denote

$$E\{(\dots)\} = \int (\dots) p_{n,\beta}(\lambda_1, \dots, \lambda_n) d\bar{\lambda}. \quad (1.3)$$

This distribution can be considered for any $\beta > 0$, but the cases $\beta = 1, 2, 4$ are especially important, since they correspond to real symmetric, hermitian, and symplectic matrix models respectively.

It is known (see [1, 7]) that if V' is a Hölder function, then the empirical spectral distribution

$$n^{-1} \sum_{j=1}^n \delta(\lambda - \lambda_j)$$

converges weakly in probability defined by (1.1) to the function ρ (equilibrium density) with a compact support σ . The density ρ maximizes the functional, defined on the class \mathcal{M}_1 of positive unit measures on \mathbb{R}

$$\mathcal{E}_V(\rho) = \max_{m \in \mathcal{M}_1} \left\{ \int \log |\lambda - \mu| dm(\lambda) dm(\mu) - \int V(\lambda) m(d\lambda) \right\} = \mathcal{E}[V]. \quad (1.4)$$

The support σ and the density ρ are uniquely defined by the conditions:

$$\begin{aligned} v(\lambda) &:= 2 \int \log |\mu - \lambda| \rho(\mu) d\mu - V(\lambda) = \sup v(\lambda) := v^*, \quad \lambda \in \sigma, \\ v(\lambda) &\leq \sup v(\lambda), \quad \lambda \notin \sigma, \quad \sigma = \text{supp}\{\rho\}. \end{aligned} \quad (1.5)$$

We are interested in the behavior of linear eigenvalue statistics, i.e.,

$$\mathcal{N}_n[h] = \sum_{j=1}^n h(\lambda_j^{(n)}), \quad (1.6)$$

In the case of smooth test function h the behavior of $\mathcal{N}_n[h]$ now is very well understood for any $\beta > 0$. It was proven in [7] that for one cut (i.e., $\sigma = [a, b]$) polynomial potentials of generic behavior and sufficiently smooth h (8 derivatives), if we consider the characteristic functional of $\mathcal{N}_n[h]$ in the form

$$\Phi_{n,\beta}[x, h] = E \left\{ e^{x(\mathcal{N}_n[h] - E\{\mathcal{N}_n[h]\})} \right\}, \quad (1.7)$$

then

$$\lim_{n \rightarrow \infty} \Phi_{n,\beta}[h] = \exp \left\{ \frac{x^2}{2\beta} (\overline{D}_\sigma h, h) \right\},$$

where the "variance operator" \overline{D}_σ and the measure ν have the form

$$\begin{aligned} (\overline{D}_\sigma h, h) &= \int_\sigma \frac{h(\lambda) d\lambda}{\pi^2 X_\sigma^{1/2}(\lambda)} \int_\sigma \frac{h'(\mu) X_\sigma^{1/2}(\mu) d\mu}{\lambda - \mu}, \\ X_\sigma(\lambda) &= (b - \lambda)(\lambda - a). \end{aligned}$$

The method of [7] was improved in [10], where it was generalized to the case of non polynomial real analytic potentials V and the test functions with 4 derivatives, and then improved once more in [16], where the case of non analytic V was also studied. The case of multi-cut (i.e. σ consisting of more than one interval) real analytic potentials was studied in [17], where it was shown that in this case fluctuations become non gaussian.

But the method, used in the case of smooth h , does not work in the case of h which have jumps. In particular, the method is not applicable to $h = 1_\Delta$, $\Delta = [a, b] \subset \sigma$, which means that $\mathcal{N}_n[h]$ is a number of eigenvalues inside the interval Δ . Moreover, it is known that for gaussian unitary and gaussian orthogonal ensembles (*GUE* and *GOE*) the variance of the eigenvalue number is proportional to $\log n$, while in the case of smooth test functions the variance is $O(1)$. Thus, it is hard to believe that the central limit theorem (CLT) for indicators can be obtained by methods similar to that for smooth test functions.

Till now there are only few results on the CLT for indicators. The case of GUE was studied a long time ago (see, e.g., [8]). In the paper [3] it was shown that the Gaussian fluctuations for GUE imply similar results for GOE and GSE (i.e., the cases when $V(\lambda) = \lambda^2/2$ and $\beta = 1$ and $\beta = 4$). Even for classical random matrix models, like the Wigner model with non gaussian entries, CLT for functions with jumps was proven (see [2]) only for the Hermitian case ($\beta = 2$), and only under the assumption that the first four moments of the entries coincide with that of GUE. There are also a number of publications where CLT for the determinantal point processes are proven (see [18] and references therein or [6]). Similar results for some special kind of Pfaffian point processes were obtained in [9].

At the present paper we use the representation of the characteristic functional of $\mathcal{N}_n[h]$ in the form of the Fredholm determinant of some operator in order to prove CLT for the indicator test functions in the case of β -models with $\beta = 1, 2, 4$. Unfortunately, since similar representations are not known for general β , the method does not work for $\beta \neq 1, 2, 4$.

Let us start from the case $\beta = 2$.

Given potential V , introduce the weight function $w_n(\lambda) = e^{-nV(\lambda)}$, and consider polynomials orthogonal on \mathbb{R} with the weight w_n i.e.,

$$\int p_l^{(n)}(\lambda) p_m^{(n)}(\lambda) w_n(\lambda) d\lambda = \delta_{l,m}. \quad (1.8)$$

It will be used below also that $\{p_l^{(n)}\}_{l=0}^n$ satisfy the recursion relation

$$\lambda p_l^{(n)}(\lambda) = a_{l+1} p_{l+1}^{(n)}(\lambda) + b_n p_l^{(n)}(\lambda) + a_l p_{l-1}^{(n)}(\lambda). \quad (1.9)$$

Then consider the orthonormalized system

$$\psi_l(\lambda) = e^{-nV(\lambda)/2} p_l^{(n)}(\lambda), \quad l = 0, \dots, \quad (1.10)$$

and construct the function

$$K_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l(\lambda) \psi_l(\mu). \quad (1.11)$$

This function is known as a reproducing kernel of the system (1.10). It is known (see, e.g., [8]) that for any x and any bounded integrable test functions h the characteristic functional defined by (1.7) for $\beta = 2$ takes the form

$$\Phi_{n,2}[x, h] = e^{-xE\{\mathcal{N}_n[h]\}} \det\left\{1 + (e^{xh} - 1)K_n\right\},$$

where the operator $(e^{xh} - 1)K_n$ has the kernel

$$((e^{xh} - 1)K_n)(\lambda, \mu) := (e^{xh(\lambda)} - 1)K_n(\lambda, \mu)$$

In particular, if $h = 1_\Delta$ and we set $x_n = x\pi/\log^{1/2} n$, then $\Phi_{n,2}[x_n, 1_\Delta]$ takes the form

$$\hat{\Phi}_{n,2}(x) := e^{-x_n E\{\mathcal{N}_n[1_\Delta]\}} E\{e^{x_n \mathcal{N}_n[1_\Delta]}\} = e^{-x_n E\{\mathcal{N}_n[1_\Delta]\}} \det\left\{1 + (e^{x_n} - 1)K_n[\Delta]\right\}, \quad (1.12)$$

where

$$K_n[\Delta](\lambda, \mu) := 1_\Delta(\lambda) K_n(\lambda, \mu) 1_\Delta(\mu). \quad (1.13)$$

Representation (1.12) allows us to prove CLT for the indicator test function in the case $\beta = 2$ (see, e.g., [13]):

Theorem 1 *Let the matrix model be defined by (1.1) with $\beta = 2$ and real analytic potential $V(\lambda) \gg \log|\lambda^2 + 1|$. Let also $\Delta = [a, b] \subset \sigma^\circ$ (here and below σ° means the internal part of the support σ of the equilibrium measure) and $x_n = x\pi \log^{-1/2} n$. Then*

$$\lim_{n \rightarrow \infty} \log \hat{\Phi}_{n,2}(x) = x^2/2. \quad (1.14)$$

Although the result is not new, its proof is an important ingredient of the proofs of CLT for the cases $\beta = 1, 4$, hence the proof is given in the beginning of Section 2.

For $\beta = 1, 4$ the situation is more complicated. It was shown in [19] that the characteristic functionals $\hat{\Phi}_{n,1}(x)$ and $\hat{\Phi}_{n,4}(x)$ can be expressed in terms of some matrix kernels (see (1.16) – (1.21) below). But the representation is less convenient than (1.8) – (1.12). It makes difficult the problems, which for $\beta = 2$ are just simple exercises.

We have

$$\begin{aligned} \hat{\Phi}_{n,1}(x) &= e^{-x_n E\{\mathcal{N}_n[1_\Delta]\}} \det^{1/2}\left\{1 + (e^{x_n} - 1)K_{n,1}[\Delta]\right\}, \\ \hat{\Phi}_{n,4}(x) &= e^{-x_n E\{\mathcal{N}_{n/2}[1_\Delta]\}} \det^{1/2}\left\{1 + (e^{x_n} - 1)K_{n,4}[\Delta]\right\}, \end{aligned} \quad (1.15)$$

where similarly to the case $\beta = 2$ the operators $K_{n,1}[\Delta]$ and $K_{n,4}[\Delta]$ are the projection on the interval Δ of some matrix operators $K_{n,1}$, $K_{n,4}$:

$$K_{n,1}[\Delta](\lambda, \mu) = 1_\Delta(\lambda)K_{n,1}(\lambda, \mu)1_\Delta(\mu), \quad K_{n,4}[\Delta](\lambda, \mu) = 1_\Delta(\lambda)K_{n,4}(\lambda, \mu)1_\Delta(\mu).$$

The matrix operators $K_{n,1}$ and $K_{n,4}$ have the form

$$K_{n,1} := \begin{pmatrix} \mathcal{S}_{n,1} & \mathcal{D}_{n,1} \\ \mathcal{I}_{n,1} - \epsilon & \mathcal{S}_{n,1}^T \end{pmatrix}, \quad \beta = 1, \quad n - \text{even}, \quad (1.16)$$

$$K_{n,4} := \frac{1}{2} \begin{pmatrix} \mathcal{S}_{n,4} & \mathcal{D}_{n,4} \\ \mathcal{I}_{n,4} & \mathcal{S}_{n,4}^T \end{pmatrix}, \quad \beta = 4, \quad (1.17)$$

where the entries are integral operators in $L_2[\mathbb{R}]$ with the kernels

$$\mathcal{S}_{n,1}(\lambda, \mu) = - \sum_{j,k=0}^{n-1} \psi_j(\lambda)(M_n^{(n)})_{jk}^{-1}(\epsilon\psi_k)(\mu), \quad \mathcal{S}_{n,1}^T(\lambda, \mu) = \mathcal{S}_{n,1}(\lambda, \mu), \quad (1.18)$$

$$\mathcal{D}_{n,1}(\lambda, \mu) = -\frac{\partial}{\partial\mu}\mathcal{S}_{n,1}(\lambda, \mu), \quad \mathcal{I}_{n,1}(\lambda, \mu) = (\epsilon\mathcal{S}_{n,1})(\lambda, \mu),$$

$$\mathcal{S}_{n/2,4}(\lambda, \mu) = - \sum_{j,k=0}^{n-1} \psi'_j(\lambda)(D_n^{(n)})_{jk}^{-1}\psi_k^{(n)}(\mu), \quad \mathcal{S}_{n/2,4}^T(\lambda, \mu) = \mathcal{S}_{n/2,4}(\lambda, \mu), \quad (1.19)$$

$$\mathcal{D}_{n,4}(\lambda, \mu) = -\frac{\partial}{\partial\mu}\mathcal{S}_{n,4}(\lambda, \mu), \quad \mathcal{I}_{n,4}(\lambda, \mu) = (\epsilon\mathcal{S}_{n,4})(\lambda, \mu),$$

$$\epsilon(\lambda - \mu) = \frac{1}{2}\text{sgn}(\lambda - \mu). \quad (1.20)$$

Here the function $\{\psi_j\}_{j=0}^n$ are defined by (1.10), sgn denotes the standard signum function, and $D_n^{(n)}$ and $M_n^{(n)}$ in (1.18) and (1.19) are the left top corner $n \times n$ blocks of the semi-infinite matrices that correspond to the differentiation operator and to some integration operator respectively.

$$\begin{aligned} D_\infty^{(n)} &:= (\psi'_j, \psi_k)_{j,k \geq 0}, \quad D_n^{(n)} = \{D_{jk}^{(n)}\}_{j,k=0}^{n-1}, \\ M_\infty^{(n)} &:= (\epsilon\psi_j, \psi_k)_{j,k \geq 0}, \quad M_n^{(n)} = \{M_{jk}^{(n)}\}_{j,k=0}^{n-1}. \end{aligned} \quad (1.21)$$

Remark 1 From the structure of the kernels it is easy to see the cases $\beta = 1, 4$ the characteristic functional can be written in the form

$$\begin{aligned} \hat{\Phi}_{n,4}(x) &= \det^{1/2} \left\{ J + (e^{x_n} - 1)\hat{A}_{n,1}[\Delta] \right\} e^{-x_n E \{\mathcal{N}_n[1_{\Delta_a}]\}}, \\ \hat{\Phi}_{n,4}(x) &= \det^{1/2} \left\{ J + (e^{x_n} - 1)\hat{A}_{n,4}[\Delta] \right\} e^{-x_n E \{\mathcal{N}_n[1_{\Delta_a}]\}}, \end{aligned}$$

where $\hat{A}_{n,1} = \mathcal{S}_{n,1}J$, $\hat{A}_{n,4} = \mathcal{S}_{n,4}J$ are skew symmetric matrices

$$(A_{n,1}[\Delta])^* = -A_{n,1}[\Delta], \quad (A_{n,4}[\Delta])^* = -A_{n,4}[\Delta], \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The main problem of studying of $\hat{\Phi}_{n,1}(x)$ and $\hat{\Phi}_{n,4}(x)$ is that the corresponding operators $K_{n,1}$ and $K_{n,4}$ (differently from the case $\beta = 2$) are not self adjoint, thus even if we know the location of eigenvalues of $K_{n,1}$ and $K_{n,4}$ we cannot say something about the location of eigenvalues of $K_{n,1}[\Delta]$ and $K_{n,4}[\Delta]$.

The idea is to prove that the eigenvalue problems for $K_{n,1}[\Delta]$ and $K_{n,4}[\Delta]$ can be reduced to the eigenvalue problem for $K_n[\Delta]$ with some finite rank perturbation. For this aim we use the result of [20], where it was observed that if V is a rational function, in particular, a polynomial of degree $2m$, then the kernels $\mathcal{S}_{n,1}, \mathcal{S}_{n,4}$ can be written as

$$\begin{aligned}\mathcal{S}_{n,1}(\lambda, \mu) &= K_n(\lambda, \mu) + n \sum_{j,k=-(2m-1)}^{2m-1} F_{jk}^{(1)} \psi_{n+j}(\lambda) \epsilon \psi_{n+k}(\mu), \\ \mathcal{S}_{n/2,4}(\lambda, \mu) &= K_n(\lambda, \mu) + n \sum_{j,k=-(2m-1)}^{2m-1} F_{jk}^{(4)} \psi_{n+j}(\lambda) \epsilon \psi_{n+k}(\mu),\end{aligned}\tag{1.22}$$

where $F_{jk}^{(1)}, F_{jk}^{(4)}$ can be expressed in terms of the matrix T_n^{-1} , where T_n is the $(2m-1) \times (2m-1)$ block in the bottom right corner of $D_n^{(n)} M_n^{(n)}$, i.e.,

$$(T_n)_{jk} := (D_n^{(n)} M_n^{(n)})_{n-2m+j, n-2m+k}, \quad 1 \leq j, k \leq 2m-1. \tag{1.23}$$

The representation was used before to study local regimes for real symmetric and symplectic matrix models. The main technical obstacle there was the problem to prove that $(T_n^{-1})_{jk}$ are bounded uniformly in n . The problem was solved initially for the case of monomial $V(\lambda) = \lambda^{2m}$ in [5], then for general one-cut real analytic V in [14] and finally for the general multi cut potential in [15], where it was shown that for generic real analytic potential V

$$|F_{jk}^{(1)}| \leq C, \quad |F_{jk}^{(4)}| \leq C. \tag{1.24}$$

To formulate the main results, let us state our conditions.

C1. V is a polynomial of degree $2m$ with a positive leading coefficient, and the support of its equilibrium measure is

$$\sigma = \bigcup_{\alpha=1}^q \sigma_\alpha, \quad \sigma_\alpha = [E_{2\alpha-1}, E_{2\alpha}] \tag{1.25}$$

C2. The equilibrium density ρ can be represented in the form

$$\rho(\lambda) = \frac{1}{2\pi} P(\lambda) \Im X^{1/2}(\lambda + i0), \quad \inf_{\lambda \in \sigma} |P(\lambda)| > 0, \tag{1.26}$$

where

$$X(z) = \prod_{\alpha=1}^{2q} (z - E_\alpha), \tag{1.27}$$

and we choose a branch of $X^{1/2}(z)$ such that $X^{1/2}(z) \sim z^q$, as $z \rightarrow +\infty$. Moreover, the function v defined by (1.5) attains its maximum only if λ belongs to σ .

Remark 2 It is known (see, e.g., [13, Theorem 11.2.4]) that for any analytic V the equilibrium density ρ always has the form (1.26) – (1.27). The function P in (1.26) is analytic and can be represented in the form

$$P(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{V'(z) - V'(\zeta)}{(z - \zeta) X^{1/2}(\zeta)} d\zeta.$$

Hence condition C2 means that ρ has no zeros in the internal points of σ and behaves like square root near the edge points. This behavior of ρ is usually called generic.

Theorem 2 Consider the matrix model (1.1) with $\beta = 1$ and even n and V satisfying conditions C1, C2. Let the interval $\Delta = [a, b] \subset \sigma^\circ$, and let the characteristic functional $\hat{\Phi}_{n,1}(x)$ be defined by (1.12) for $\beta = 1$ with $x_n = x\pi \log^{-1/2} n$. Then

$$\lim_{n \rightarrow \infty} \log \hat{\Phi}_{n,1}(x) = x^2.$$

Theorem 3 Consider the matrix model (1.1) with $\beta = 4$ and V satisfying conditions C1, C2. Let the interval $\Delta = [a, b] \subset \sigma^\circ$ and let characteristic functional $\hat{\Phi}_{n,4}(x)$ be defined by (1.12) for $\beta = 4$ with $x_n = x\pi \log^{-1/2} n$. Then

$$\lim_{n \rightarrow \infty} \log \hat{\Phi}_{n,4}(x) = x^2/4.$$

2 Proofs

Proof of Theorem 1 Set $F(x_n) := \log \hat{\Phi}_{n,2}(x)$ and consider the Taylor expansion of $F(x_n)$ with respect to x_n up to the second order

$$\begin{aligned} F(x_n) &= \frac{x_n^2}{2} \text{Tr } K_n[\Delta](1 - K_n[\Delta]) + \frac{x_n^3}{6} \text{Tr } K_n[\Delta](1 - K_n[\Delta])\tilde{R}(K_n[\Delta]), \\ C_1 &\leq \tilde{R}(t) \leq C_2, \quad t \in [0, 1]. \end{aligned} \quad (2.1)$$

Lemma 1

$$\text{Tr } K_n[\Delta](1 - K_n[\Delta]) = \int_{\Delta} d\lambda \int_{\bar{\Delta}} K_n^2(\lambda, \mu) d\mu = \pi^{-2} \log n(1 + o(1)). \quad (2.2)$$

The lemma implies that the first term in the r.h.s. of (2.1) tends to $x^2/2$, while the second one is bounded by $cx_n^3 \log n = o(1)$, since

$$C_1 \text{Tr } K_n[\Delta](1 - K_n[\Delta]) \leq \text{Tr } K_n[\Delta](1 - K_n[\Delta])\tilde{R}(K_n[\Delta]) \leq C_2 \text{Tr } K_n[\Delta](1 - K_n[\Delta]).$$

Hence, we get the assertion of Theorem 1. Thus, we are left to prove Lemma 1

Proof of Lemma 1. Take $d_n = \log^{1/3} n$ and write

$$\begin{aligned} \text{Tr } K_n[\Delta](1 - K_n[\Delta]) &= \int_{\Delta} d\lambda \int_{\bar{\Delta}} d\mu K_n^2(\lambda, \mu) = \left(\int_a^{a+d_n} d\lambda + \int_{a+d_n}^{b-d_n} d\lambda + \int_{b-d_n}^b d\lambda \right) \\ &\quad \times \left(\int_{a-d_n}^a d\mu + \int_b^{b+d_n} d\mu + \int_{-\infty}^{a-d_n} d\mu + \int_{b+d_n}^{\infty} d\mu \right) K_n^2(\lambda, \mu). \end{aligned} \quad (2.3)$$

The Christoffel-Darboux formula implies

$$\int K_n^2(\lambda, \mu)(\lambda - \mu)^2 d\lambda d\mu = a_n \int (\psi_n(\lambda)\psi_{n-1}(\mu) - \psi_n(\mu)\psi_{n-1}(\lambda))^2 d\lambda d\mu = 2a_n \leq C,$$

where a_n is the recursion coefficient of (1.9), and we have used the result of [11] (see also [13], Chapter, Lemma) on the uniform boundedness of a_n , as $n \rightarrow \infty$. Then

$$\int_{d_n \leq |\lambda - \mu|} K_n^2(\lambda, \mu) d\lambda d\mu \leq C d_n^{-2} = O(\log^{2/3} n),$$

which implies that

$$\begin{aligned} \text{Tr } K_n[\Delta](1 - K_n[\Delta]) &= \int_a^{a+d_n} d\lambda \int_{a-d_n}^a d\mu K_n^2(\lambda, \mu) \\ &\quad + \int_{b-d_n}^b d\lambda \int_b^{b+d_n} d\mu K_n^2(\lambda, \mu) + O(\log^{2/3} n) = I_a + I_b + O(\log^{2/3} n). \end{aligned} \quad (2.4)$$

To find I_a , we apply the results of [4], according to which for λ, μ from the bulk of the spectrum the reproducing kernel has the form

$$\begin{aligned} K_n(\lambda, \mu) &= h(\lambda, \mu) \frac{\sin n\pi(\phi(\lambda) - \phi(\mu))}{\pi(\lambda - \mu)} (1 + O(n^{-1})) \\ &\quad + \sum_{\pm} r_{\pm, \pm}(\lambda, \mu) e^{i\pi n(\pm\phi(\lambda) \pm \phi(\mu))}, \end{aligned} \quad (2.5)$$

where h and ϕ for (λ, μ) in the bulk of the spectrum are smooth, positive, bounded from both sides functions, the remainder functions $r_{+,+}$, $r_{-,-}$, $r_{+,-}$, $r_{-,+}$ have uniformly bounded derivatives in both variables, and \sum_{\pm} means the summation with respect to all combinations of signs in the exponents. Moreover,

$$\begin{aligned}\phi'(\lambda) &> c_0, \quad \text{if } |\lambda - E_k| \geq \tilde{\varepsilon}, \quad k = 1, \dots, 2q, \\ h(\lambda, \lambda) &= 1.\end{aligned}$$

It is easy to see that the remainder terms in the r.h.s. of (2.5) after integration in the limits, written in the r.h.s. of (2.4), give us at most $O(d_n^2)$. Hence, we need only to find the contribution of the first term of (2.5). Performing the change of variables $\lambda = a + x/(n\phi'(a))$, $\mu = a - y/(n\phi'(a))$, we get

$$\begin{aligned}I_a &= \int_0^{nd_n} dx \int_0^{nd_n} dy (1 + o(1)) \frac{\sin^2(\pi(x+y)(1+o(1)))}{\pi^2(x+y)^2} dx dy + O(d_n^2) \\ &= \int_0^{nd_n} dx \int_0^{nd_n} dy \frac{\sin^2(\pi(x+y))}{\pi^2(x+y)^2} dx dy + o(\log nd_n) + O(d_n^2) \\ &= \left(\int_0^1 dx + \int_1^{nd_n} dx \right) \left(\int_0^1 dy + \int_1^{nd_n} dy \right) \frac{\sin^2(\pi(x+y))}{\pi^2(x+y)^2} dy + o(\log nd_n) + O(d_n^2) \\ &= \int_1^{nd_n} dx \int_1^{nd_n} dy \frac{1 - \cos 2\pi(x+y)}{2\pi^2(x+y)^2} + O(1) + o(\log nd_n) + O(d_n^2) \\ &= \int_1^{nd_n} dx \int_1^{\infty} dy \frac{1}{2\pi^2(x+y)^2} + O(1) + o(\log nd_n) = \frac{1}{2\pi^2} \log n(1 + o(1)).\end{aligned}$$

Similarly

$$I_b = \frac{1}{2\pi^2} \log n(1 + o(1)).$$

Then in view of (2.4) we obtain (2.2).

□

Proof of Theorem 2

Let us consider the eigenvalue problem for $\hat{K}_{n,1}[\Delta]$:

$$\begin{cases} \mathcal{S}_{n,1}f_{\Delta} + \mathcal{D}_{n,1}g_{\Delta} = Ef_{\Delta}, \\ \mathcal{I}_{n,1}f_{\Delta} - \epsilon f_{\Delta} + \mathcal{S}_{n,1}^T g_{\Delta} = Ef_{\Delta}. \end{cases} \quad (2.6)$$

Here and below

$$f_{\Delta} = 1_{\Delta}f, \quad g_{\Delta} = 1_{\Delta}g.$$

Observe, that since all the functions in the first line of (2.6) are analytic, the equation is valid also outside of Δ . Apply the operator ϵ to both sides of the equation. We get

$$\begin{aligned} &\begin{cases} \mathcal{I}_{n,1}f_{\Delta} + \mathcal{S}_{n,1}^T g_{\Delta} = E\epsilon f_{\Delta} + E\epsilon f_{\bar{\Delta}}, \\ \mathcal{I}_{n,1}f_{\Delta} - \epsilon f_{\Delta} + \mathcal{S}_{n,1}^T g_{\Delta} = Eg_{\Delta}, \end{cases} \\ &\Rightarrow Eg_{\Delta} = (E-1)(\epsilon f_{\Delta}) + 1_{\Delta}E\epsilon f_{\bar{\Delta}}, \end{aligned} \quad (2.7)$$

where we use that integration by parts gives us that $\epsilon \mathcal{D}_{n,1} = \mathcal{S}_{n,1}^T$, and denote

$$f_{\bar{\Delta}} = f - f_{\Delta}$$

with $\bar{\Delta}$ being a complement of Δ . Observe that

$$\epsilon f_{\bar{\Delta}}(\lambda) = \frac{1}{2} \int_{-\infty}^a f(t) dt - \frac{1}{2} \int_b^{\infty} f(t) dt =: (f, \Psi_{\Delta}) = \text{const}, \quad \lambda \in \Delta. \quad (2.8)$$

Multiply the first line of (2.7) by E and use the above equation for Eg_Δ . Integration by parts gives us

$$\begin{aligned}\mathcal{D}_{n,1}(\epsilon f_\Delta)(\lambda) &= \mathcal{S}_{n,1}f_\Delta(\lambda), \\ \mathcal{D}_{n,1}(\mathbf{1}_\Delta \epsilon f_\Delta)(\lambda) &= (\mathcal{S}_{n,1}(\lambda, a) - \mathcal{S}_{n,1}(\lambda, b))(f, \Psi_\Delta) =: Pf,\end{aligned}\tag{2.9}$$

where by the above definition, P is a rank one operator in $L_2[\Delta]$. We obtain

$$(2E - 1)\mathcal{S}_{n,1}[\Delta]f_\Delta - E^2 f_\Delta + EPf = 0.$$

Hence the solutions $\{E_k\}$ of (2.6) are solutions of the equation

$$\mathcal{P}(E) := \det\{E^2 - (2E - 1)\mathcal{S}_{n,1}[\Delta] + EP\} = 0.$$

It is evident that $\mathcal{P}(E)$ is a polynomial of $2n$ th degree, and E_k are the roots of $\mathcal{P}(E)$. We are interested in

$$\prod_{k=1}^{2n} (1 + \delta_n E_k) = \delta_n^{2n} \prod_{k=1}^{2n} (\delta_n^{-1} + E_k) = \delta_n^{2n} \mathcal{P}(-\delta_n^{-1}),$$

where $\delta_n := e^{x_n} - 1$. Thus we obtain

$$\log \hat{\Phi}_{n,1}(x) = -x_n E \{\mathcal{N}_n[\mathbf{1}_\Delta]\} + \frac{1}{2} \log \det \left\{ 1 + (2\delta_n + \delta_n^2) \mathcal{S}_{n,1}[\Delta] + \delta_n P \right\}.\tag{2.10}$$

Now we use (1.22). Substituting the representation in (2.10) we get

$$\begin{aligned}\log \hat{\Phi}_{n,1}(x) &= -x_n E \{\mathcal{N}_n[\mathbf{1}_\Delta]\} + \frac{1}{2} \log \det \left\{ 1 + (2\delta_n + \delta_n^2) K_n(\Delta) \right\} \\ &\quad + \frac{1}{2} \log \det \left\{ (1 + R(\delta_n P_1 + \tilde{\delta}_n n \sum_{k,j=-(2m-1)}^{2m-1} F_{kj}^{(1)} Q_{kj})), \right\},\end{aligned}\tag{2.11}$$

where $\{Q_{kj}\}$ are rank one operators with the kernels $Q_{kj}(\lambda, \mu) = \psi_{n+k}(\lambda) \epsilon \psi_{n+j}(\mu)$,

$$R = (1 - \tilde{\delta}_n K_n[\Delta])^{-1}, \quad \tilde{\delta}_n = (e^{2x_n} - 1) = 2\delta_n + \delta_n^2.$$

According to the standard linear algebra argument

$$\det(1 + \tilde{\delta}_n \sum a_i \otimes b_i) = \det\{\delta_{ij} + \tilde{\delta}_n(a_i, b_j)\}.$$

Taking into account the formula and the structure of the remainder in (2.11), we conclude that in order to prove that the last term in (2.11) is small, it suffices to prove that

$$\begin{aligned}|n(R\psi_{n-j}, \epsilon \psi_{n+k})| &\leq C\delta_n, \\ |(R\mathbf{1}_\Delta S_n(x, a), \Psi_\Delta)| &\leq C\delta_n, \quad |(R\mathbf{1}_\Delta S_n(x, b), \Psi_\Delta)| \leq C\delta_n.\end{aligned}\tag{2.12}$$

The last two inequalities are trivial, since

$$\text{supp } \Psi_\Delta = \bar{\Delta}, \quad \text{supp } R\mathbf{1}_\Delta S_n(x, a) = \text{supp } R\mathbf{1}_\Delta S_n(x, b) = \Delta.$$

For the proof of the first inequality of (2.12) we need the following lemma.

Lemma 2 *Set*

$$v_n(\lambda) := \mathbf{1}_{\bar{\Delta}}(\lambda) \int_{\Delta} d\mu K_n(\lambda, \mu).\tag{2.13}$$

Then for any $\lambda \in \bar{\Delta}$

$$|v_n(\lambda)| \leq \frac{C}{1 + n|\lambda - b|} + \frac{C}{1 + n|\lambda - a|}.\tag{2.14}$$

The proof of the lemma is given after the proof of Theorem 2. Now we continue the proof of (2.12). The first bound of (2.12) is a corollary of three estimates

$$\begin{aligned} |n(1_\Delta \psi_{n+k}, \epsilon \psi_{n+j})| &\leq C\delta_n, \quad |n(\psi_{n+k}, K_n[\Delta] \epsilon \psi_{n+j})| \leq C\delta_n, \\ \|n(K_n[\Delta] - K_n^2[\Delta]) \epsilon \psi_{n+j}\| &\leq C. \end{aligned} \quad (2.15)$$

Indeed, the third bound of (2.15) yields for $m \geq 2$

$$\begin{aligned} \|n(K_n^m[\Delta] - K_n[\Delta]) \epsilon \psi_{n+j}\| &\leq \sum_{l=0}^{m-1} \|K_n^l[\Delta] n(K_n^2[\Delta] - K_n[\Delta]) \epsilon \psi_{n+j}\| \\ &\leq m \|n(K_n^2[\Delta] - K_n[\Delta]) \epsilon \psi_{n+j}\| \leq mC. \end{aligned}$$

Here we used also that $\|K_n[\Delta]\| \leq 1$. Thus,

$$\begin{aligned} \left\| \sum_{m=2}^{\infty} \tilde{\delta}_n^m n(K_n^m[\Delta] - K_n[\Delta]) \epsilon \psi_{n+j} \right\| &\leq \sum_{m=2}^{\infty} mC \tilde{\delta}_n^m \leq C\tilde{\delta}_n^2, \\ \Rightarrow \left\| \sum_{m=2}^{\infty} n\tilde{\delta}_n^m K_n^m[\Delta] \epsilon \psi_{n+j} - \frac{n(2\tilde{\delta}_n^2 - \tilde{\delta}_n^3)}{(1 - \tilde{\delta}_n)^2} K_n[\Delta] \epsilon \psi_{n+j} \right\| &\leq C\tilde{\delta}_n^2. \end{aligned}$$

Combining this inequality with the first two bounds of (2.15) we obtain the first bound of (2.12).

To prove (2.15), we use the result of [15, Lemma 2], according to which

$$\epsilon \psi_{n+j} = n^{-1/2} c_{n+j} + O(n^{-1}),$$

where c_{n+j} is some constant, bounded uniformly in n . Using this fact, we conclude that to prove (2.15) it suffices to prove that

$$\begin{aligned} |n^{1/2}(1_\Delta \psi_{n+k}, 1_\Delta)| &\leq C\delta_n, \quad |n^{1/2}(K_n[\Delta] \psi_{n+k}, 1_\Delta)| \leq C\delta_n, \\ \|n^{1/2}(K_n[\Delta] - K_n^2[\Delta]) 1_\Delta\| &\leq C. \end{aligned} \quad (2.16)$$

Lemma 2 yields

$$\begin{aligned} \|(K_n[\Delta] - K_n^2[\Delta]) 1_\Delta\| &= \left\| \int_{\bar{\Delta}} d\nu \int_{\Delta} d\mu K_n(\lambda, \nu) K_n(\mu, \nu) \right\| \\ &= \|K_n v_n\| \leq \|v_n\| \leq C_1 n^{-1/2}, \end{aligned}$$

hence we obtain the last inequality of (2.16). The first inequality of (2.16) is a simple corollary of the result of [4, Theorem 1.1], according to which

$$\psi_{n+k}(\lambda) = R_k(\lambda) \cos(n\pi\phi(\lambda) + m_k(\lambda)) (1 + O(n^{-1}))$$

where R_k and m_k are smooth functions. Using this result, we can integrate by parts and obtain the first inequality of (2.16) (even with $Cn^{-1/2}$ in the r.h.s. instead of $C\delta_n$). In addition, since

$$K_n \psi_{n+k} = 1_{k < 0} \psi_{n+k},$$

we have

$$K_n[\Delta] \psi_{n+k} = 1_{k < 0} 1_\Delta \psi_{n+k} - K_n 1_{\bar{\Delta}} \psi_{n+k}.$$

The bound of the first term is given by the first inequality of (2.16). For the second term write

$$|(K_n 1_{\bar{\Delta}} \psi_{n+k}, 1_\Delta)| = |(1_{\bar{\Delta}} \psi_{n+k}, K_n 1_\Delta)| = |(1_{\bar{\Delta}} \psi_{n+k}, v_n)| \leq Cn^{-1/2}.$$

Hence, we complete the proof of the second inequality of (2.16).

It was explained above that (2.16) imply (2.15), which combined with (2.11) yields

$$\log \hat{\Phi}_{n,1}(x) = -x_n E\{\mathcal{N}_n[\mathbf{1}_\Delta]\} + \frac{1}{2} \log \det \left\{ 1 + (e^{2x_n} - 1) K_n[\Delta] \right\} + O(\delta_n).$$

Then similarly to the case $\beta = 2$ we have

$$\begin{aligned} \frac{1}{2} \text{Tr} \log(1 + (e^{2x_n} - 1) K_n[\Delta]) &= x_n E\{\mathcal{N}_n[\mathbf{1}_\Delta]\} + x_n^2 \text{Tr} K_n[\Delta](1 - K_n[\Delta]) \\ &\quad + \frac{(2x_n)^3}{12} \text{Tr} K_n[\Delta](1 - K_n[\Delta]) \tilde{R}(K_n[\Delta]). \end{aligned} \quad (2.17)$$

By Lemma 1,

$$\text{Tr} K_n[\Delta](1 - K_n[\Delta]) = \pi^{-2} \log n (1 + o(1)),$$

hence the limit of the second term of (2.17) is x^2 and the last term is $O(\log^{-1/2} n)$.

□

Proof of Lemma 2 The proof is based on the representation (2.5). Integrating by parts, it is easy to see that the contribution of the remainder terms (written in \sum_{\pm}) is at most $O(n^{-1})$. Hence we need to consider only the contribution of the first term in the r.h.s. of (2.5). Take $\lambda < a$ and consider the change of variables $x = \phi(a) - \phi(\lambda)$, $y = \phi(\mu) - \phi(a)$. Let φ be the inverse function of $\phi(x) - \phi(a)$ and $a - \lambda = \Delta_\lambda \geq 0$. Then the main part of our integral takes the form

$$\begin{aligned} F(\lambda) &:= \int_0^d dy \varphi'(y) \tilde{h}_\lambda(y) \frac{\sin n\pi(x+y)}{\varphi(y) + \Delta_\lambda} \\ &= \varphi'(0) \tilde{h}_\lambda(0) \int_0^d dy \frac{\sin n\pi(x+y)}{\varphi(y) + \Delta_\lambda} + O(n^{-1}) \\ &= C \int_0^{nd} dy' \frac{\sin \pi(nx + y')}{n\varphi(y'/n) + n\Delta_\lambda} + O(n^{-1}), \end{aligned} \quad (2.18)$$

where

$$d = \phi(b) - \phi(a), \quad \tilde{h}_\lambda(y) = h(\lambda, \varphi(y)),$$

and we have used the fact that the function

$$\frac{\varphi'(y) \tilde{h}_\lambda(y) - \varphi'(0) \tilde{h}_\lambda(0)}{y + \Delta_\lambda}$$

has a bounded derivative, hence integration by parts with $\sin n\pi(x+y)$ gives us $O(n^{-1})$

$$\begin{aligned} F(\lambda) &:= C \int_0^{1-\{nx\}} dy \frac{\sin \pi(\{nx\} + y)}{n\varphi(y'/n) + n\Delta_\lambda} - C \sum_{k=1}^{nd} \left(\int_{-\{nx\}}^{1-\{nx\}} dy \frac{\sin \pi(\{x\} + y)}{n\varphi((y' + k)/n) + n\Delta_\lambda} \right. \\ &\quad \left. - \int_{-\{nx\}}^{1-\{nx\}} \frac{\sin \pi(\{nx\} + y')}{n\varphi((y' + k + 1)/n) + n\Delta_\lambda} dy' \right). \end{aligned}$$

Observe that the series above is of alternating sign, and modules of the terms decay, as k grows (recall that $\varphi(y)$ is an increasing function of y). Thus,

$$\begin{aligned} F(\lambda) &\geq C \int_0^{1-\{nx\}} dy' \frac{\sin \pi(\{nx\} + y')}{n\varphi(y'/n) + n\Delta_\lambda} - C \int_{-\{nx\}}^{1-\{nx\}} dy \frac{\sin \pi(\{nx\} + y)}{n\varphi((y' + 1)/n) + n\Delta_\lambda}, \\ F(x) &\leq C \int_0^{1-\{nx\}} dy \frac{\sin \pi(\{nx\} + y)}{(x + y)} - C \int_{-\{nx\}}^{1-\{nx\}} dy' \frac{\sin \pi(\{nx\} + y')}{n\varphi((y' + 1)/n) + n\Delta_\lambda} \\ &\quad + C \int_{-\{nx\}}^{1-\{nx\}} dy' \frac{\sin \pi(\{nx\} + y')}{n\varphi((y' + 2)/n) + n\Delta_\lambda}. \end{aligned}$$

These bounds combined with (2.18) prove (2.14) for $\lambda < a$. For $\lambda > b$ the proof is the same.

□

Proof of Theorem 3 The proof is very similar to that of Theorem 2, hence we present it very briefly. We consider the eigenvalue problem for $\hat{K}_{n,4}[\Delta]$ ($\beta = 1$).

$$\begin{cases} \mathcal{S}_{n,4}f_{\Delta} + \mathcal{D}_{n,4}g_{\Delta} = Ef_{\Delta}, \\ \mathcal{I}_{n,4}f_{\Delta} + \mathcal{S}_{n,4}^Tg_{\Delta} = Ef_{\Delta}. \end{cases} \quad (2.19)$$

Apply the operator ϵ to both sides of the first equation and then subtract the second line from the first. We get

$$Eg_{\Delta} = E(\epsilon f_{\Delta}) + \mathbf{1}_{\Delta} E\epsilon f_{\Delta}.$$

Substituting the relation in the first line of (2.19), we obtain

$$2\mathcal{S}_{n,4}[\Delta]f_{\Delta} - Ef_{\Delta} + Pf = 0,$$

where (cf (2.9))

$$Pf := (\mathcal{S}_{n,4}(\lambda, a) - \mathcal{S}_{n,4}(\lambda, b))(f, \Psi_{\Delta})$$

is a rank one operator. Taking into account (1.19) and (1.15), we have now (cf (2.10))

$$\log \hat{\Phi}_{n,4}(x) = -x_n E\{\mathcal{N}_n[\mathbf{1}_{\Delta}]\} + \frac{1}{2} \log \det \left\{ 1 + \delta_n \mathcal{S}_{n,4} \Delta + \frac{\delta_n}{2} P \right\}.$$

Applying (1.22) and repeating the argument used in the proof of Theorem 2, we obtain the assertion of Theorem 3.

□

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